Coincidences in 4 dimensions

Michael Baake and Peter Zeiner*
Faculty of Mathematics, Bielefeld University, Box 100131, 33501 Bielefeld, GERMANY

February 2, 2008

Abstract

The coincidence site lattices (CSLs) of prominent 4-dimensional lattices are considered. CSLs in 3 dimensions have been used for decades to describe grain boundaries in crystals. Quasicrystals suggest to also look at CSLs in dimensions $d > 3$. Here, we discuss the CSLs of the root lattice A_4 and the hypercubic lattices, which are of particular interest both from the mathematical and the crystallographic viewpoint. Quaternion algebras are used to derive their coincidence rotations and the CSLs. We make use of the fact that the CSLs can be linked to certain ideals and compute their indices, their multiplicities and encapsulate all this in generating functions in terms of Dirichlet series. In addition, we sketch how these results can be generalised for 4–dimensional $\mathbb{Z}$–modules by discussing the icosian ring.

1 Introduction

Let us begin with a brief recapitulation of concepts and methods.

1.1 Coincidence site lattices and modules

Let $\varGamma \subseteq \mathbb{R}^d$ be a d-dimensional lattice and $R \in \mathrm{O}(d)$ an isometry. Then, R is called a *coincidence isometry* of $\varGamma$ if $\varGamma(R) := \varGamma \cap R\varGamma$ is a lattice of finite index in $\varGamma$, and $\varGamma(R)$ is called an (ordinary or simple) *coincidence site lattice (CSL)* (for an introduction, see [1]). The group of all coincidence isometries of $\varGamma$ is called $OC(\varGamma)$, whereas the subgroup of all orientation preserving isometries is called the group of coincidence rotations and referred to as $SOC(\varGamma)$. The *coincidence index* $\varSigma(R)$ is defined as the index of $\varGamma(R)$ in $\varGamma$, and $\sigma = \{\varSigma(R) \mid R \in OC(\varGamma)\}$ denotes the (ordinary or simple) coincidence spectrum. These definitions can be generalised to include the possibility of multiple CSLs, see [2, 3, 4].

All these concepts also extend to $\mathbb{Z}$–modules [1, 5]. By a $\mathbb{Z}$–module of dimension d and rank k, we mean the $\mathbb{Z}$–span of k rationally independent vectors in $\mathbb{R}^d$.

1.2 Quaternions and rotations

Rotations in $\mathbb{R}^4$ can be parameterised by a pair of quaternions [6, 7, 8]. Recall that the quaternion algebra $\mathbb{H}(\mathbb{R})$ is the vector space $\mathbb{R}^4$ equipped with a special (associative but non–commutative) product. In standard notation, where $1, i, j, k$ form an orthonormal basis of $\mathbb{R}^4$, the product satisfies the defining relations[1] $i^2 = j^2 = k^2 = -1$ and $ijk = -kji = -1$. This definition is extended to all quaternions by linearity. All non–zero quaternions $q = (a, b, c, d) = a + bi + cj + dk$ have an inverse $q^{-1} = \frac{1}{|q|^2}\bar{q}$, where $\bar{q} = (a, -b, -c, -d)$ is the conjugate of q and $|q|^2 = a^2 + b^2 + c^2 + d^2$ is its (reduced) norm. With this notation at hand, we can write any rotation in $\mathbb{R}^4$ as

$$Rx = R(q, p)x = \frac{1}{|qp|} qx\bar{p}. \tag{1}$$

**Corresponding author. Email: pzeiner@math.uni-bielefeld.de

[1] Note that these relations are very similar to the relations satisfied by the Pauli matrices. In fact, dividing the Pauli matrices by the imaginary unit gives a representation of i, j, k by 2×2–matrices.

Reflections and, more generally, all orientation reversing isometries may be parameterised by a pair of quaternions as well, but a conjugation is required in addition. For our purposes, it is sufficient to focus the discussion to (proper) rotations. This is no restriction, since all the lattices and modules we discuss have a reflection as a symmetry operation and thus every orientation reversing coincidence isometry can be written as the product of this particular reflection and a coincidence rotation with the same index. In other words, every orientation reversing coincidence isometry is symmetry related to a coincidence rotation with the same index (see below). Hence all CSLs are already obtained by (proper) coincidence rotations and the spectrum does not change either. There are exactly as many coincidence rotations as orientation reversing isometries, and the total number of coincidence isometries is just twice the number of coincidence rotations.

1.3 Symmetry related rotations and CSLs

Two coincidence rotations R and R' are called *symmetry related* if there is an element Q in the point group $P(\varGamma)$ such that $R' = RQ$. Note that $R\varGamma = R'\varGamma$ holds if and only if R and R' are symmetry related. In particular, two symmetry related rotations R and R' generate the same CSL, so that $\varGamma(R) = \varGamma(R')$. However, the converse statement is not true in general, i.e., two rotations generating the same CSL need *not* be symmetry related. In particular, in all the examples presented below, there are additional rotations that generate the same CSL.

2 Hypercubic lattices

The hypercubic lattices have been discussed in detail in [1, 9]. Here, we just review the most important facts, reformulate some of them and add some new details. In particular, we present an explicit expression for the number of CSLs.

2.1 Centred hypercubic lattice

The centred hypercubic lattice D_4^*, which is the dual lattice of the root lattice D_4, which in turn is a similar sublattice of D_4^*, can be identified with the Hurwitz ring $\mathbb{J}$ of integer quaternions. This is the $\mathbb{Z}$–span (meaning the set of all integral linear combinations) of the quaternions $(1,0,0,0), (0,1,0,0), (0,0,1,0), \frac{1}{2}(1,1,1,1)$. Any coincidence rotation can be parameterised by a pair of integer quaternions. More precisely, a rotation is a coincidence rotation of D_4^* if and only if it can be parameterised by an admissible pair of quaternions [1, 9]. Here, we call a quaternion $q \in \mathbb{J}$ *primitive* if $\frac{1}{n}q \in \mathbb{J}$ with $n \in \mathbb{N}$ implies $n = 1$. A primitive pair (q,p) is called *admissible* if $|qp| \in \mathbb{Z}$. In addition, we call a primitive quaternion $q \in \mathbb{J}$ *reduced* if $|q|^2$ is odd (for a more general discussion of reduced quaternions, see [10]). Due to the unique (left or right) prime decomposition in $\mathbb{J}$, we can decompose any primitive quaternion $q \in \mathbb{J}$ as $q_r s$, where q_r is reduced and $|s|^2$ is a power of 2. Note that q_r is unique up to right multiplication by a unit[2] of $\mathbb{J}$.

For any admissible pair (q,p), we can define the factors

$$\alpha_q := \sqrt{\frac{|p_r|^2}{\gcd(|q_r|^2, |p_r|^2)}} \qquad \text{and} \qquad \alpha_p := \sqrt{\frac{|q_r|^2}{\gcd(|q_r|^2, |p_r|^2)}}.$$

The pair $(q_\alpha, p_\alpha) := (\alpha_q q_r, \alpha_p p_r)$ is called the *reduced extension* of the pair (q,p). Note that two admissible pairs are symmetry related if and only if their reduced extension pairs are equal (up to units). The coincidence index of a rotation $R(q,p)$ can now be written [1, 9] as

$$\varSigma_{D_4}(R(q,p)) = \operatorname{lcm}(|q_r|^2, |p_r|^2) = |q_\alpha|^2 = |p_\alpha|^2. \tag{2}$$

Since any (positive) integer can be written as a sum of four squares, this implies that the spectrum σ is the set of all odd natural numbers, $\sigma = \{1, 3, 5, 7, \ldots\}$

The CSLs can be calculated explicitly [11]. They read

$$\mathbb{J} \cap \frac{q\mathbb{J}\bar{p}}{|qp|} = q_\alpha \mathbb{J} + \mathbb{J}\bar{p}_\alpha. \tag{3}$$

[2] Recall that the units are those quaternions $u \in \mathbb{J}$ for which $|u|^2 = 1$.

Since the point group of D_4 contains 576 rotations, the number of coincidence rotations of a given index n can be written as $576 f_{D_4}^{rot}(n)$. This gives a multiplicative arithmetic function, i.e.

$$f_{D_4}^{rot}(mn) = f_{D_4}^{rot}(m) f_{D_4}^{rot}(n) \qquad \text{if } m, n \text{ are coprime,} \tag{4}$$

which is completely determined by $f_{D_4}^{rot}(1) = 1$, $f_{D_4}^{rot}(2^r) = 0$ for $r \geq 1$ and

$$f_{D_4}^{rot}(p^r) = \frac{p+1}{p-1} p^{r-1}(p^{r+1} + p^{r-1} - 2) \quad \text{if } p \text{ is an odd prime, } r \geq 1. \tag{5}$$

The multiplicativity of $f_{D_4}^{rot}(n)$ suggests the use of a Dirichlet series as a generating function for it:

$$\begin{aligned} \Phi_{D_4}^{rot}(s) &= \sum_{n=1}^{\infty} \frac{f_{D_4}^{rot}(n)}{n^s} = \prod_{p \neq 2} \frac{(1+p^{-s})(1+p^{1-s})}{(1-p^{1-s})(1-p^{2-s})} \\ &= 1 + \frac{16}{3^s} + \frac{36}{5^s} + \frac{64}{7^s} + \frac{168}{9^s} + \frac{144}{11^s} + \frac{196}{13^s} + \frac{576}{15^s} + \frac{324}{17^s} + \cdots . \end{aligned} \tag{6}$$

Similarly, one can calculate the number $f_{D_4}(n)$ of different CSLs of a given index n. Clearly, one has $f_{D_4}(n) \leq f_{D_4}^{rot}(n)$, and we do not have equality in general, since two CLSs that are not symmetry related may generate the same CSL. In fact, we have the following theorem which tells us when two CSLs are equal [11].

Theorem 1 *Let (q_1, p_1) and (q_2, p_2) be two primitive reduced admissible pairs. Then,*

$$\mathbb{J} \cap \frac{q_1 \mathbb{J} \bar{p}_1}{|q_1 p_1|} = \mathbb{J} \cap \frac{q_2 \mathbb{J} \bar{p}_2}{|q_2 p_2|} \tag{7}$$

holds if and only if $|q_1 p_1| = |q_2 p_2|$, $\mathrm{lcm}(|q_1|^2, |p_1|^2) = \mathrm{lcm}(|q_2|^2, |p_2|^2)$, $\mathrm{glcd}(q_1, |p_1 q_1|) = \mathrm{glcd}(q_2, |p_2 q_2|)$ *and* $\mathrm{glcd}(p_1, |p_1 q_1|) = \mathrm{glcd}(p_2, |p_2 q_2|)$ *hold.*

Here, glcd denotes the greatest left common divisor in $\mathbb{J}$, which is defined up to units of $\mathbb{J}$. We find that $f_{D_4}(n)$ is multiplicative, too, and is determined by $f_{D_4}(1) = 1$, $f_{D_4}(2^r) = 0$ for $r \geq 1$ and

$$f_{D_4}(p^r) = \begin{cases} \frac{(p+1)^2}{p^3-1} \left(p^{2r+1} + p^{2r-2} - 2p^{(r-1)/2}\right), & \text{if } r \geq 1 \text{ is odd,} \\ \frac{(p+1)^2}{p^3-1} (p^{2r+1} + p^{2r-2} - 2p^{r/2-1} \frac{1+p^2}{1+p}), & \text{if } r \geq 2 \text{ is even,} \end{cases} \tag{8}$$

for odd primes p. The corresponding Dirichlet series reads

$$\begin{aligned} \Phi_{D_4}(s) &= \sum_{n=1}^{\infty} \frac{f_{D_4}(n)}{n^s} = \prod_{p \neq 2} \frac{1 + p^{-s} + 2p^{1-s} + 2p^{-2s} + p^{1-2s} + p^{1-3s}}{(1-p^{2-s})(1-p^{1-2s})} \\ &= 1 + \frac{16}{3^s} + \frac{36}{5^s} + \frac{64}{7^s} + \frac{152}{9^s} + \frac{144}{11^s} + \frac{196}{13^s} + \frac{576}{15^s} + \frac{324}{17^s} + \cdots . \end{aligned}$$

Differences to Eq. (6) occur for all integers that are divisible by the square of an odd prime.

2.2 Primitive hypercubic lattice

It is well known that the primitive hypercubic lattice $\mathbb{Z}^4$, which is a sublattice of D_4^* of index 2, has a smaller point group than D_4^*, containing only 192 rotations, so that $[Aut(D_4^*) : Aut(\mathbb{Z}^4)] = 3$. As a consequence, every class of symmetry related rotations splits into three classes, one of which has the same coincidence index as before, $\Sigma_{\mathbb{Z}^4}(R) = \Sigma_{D_4}(R)$, while the other two classes have index $2\Sigma_{D_4}(R)$. The number of coincidence rotations is thus given by $192 f_{\mathbb{Z}^4}^{rot}(n)$, where $f_{\mathbb{Z}^4}^{rot}(n)$ is again multiplicative, but slightly more complicated than $f_{D_4}^{rot}(n)$ (see [1, 9]). It has the generating function

$$\begin{aligned} \Phi_{\mathbb{Z}^4}^{rot}(s) &= \sum_{n=1}^{\infty} \frac{f_{\mathbb{Z}^4}^{rot}(n)}{n^s} = (1+2^{1-s}) \Phi_{D_4}^{rot}(s) = (1+2^{1-s}) \prod_{p \neq 2} \frac{(1+p^{-s})(1+p^{1-s})}{(1-p^{1-s})(1-p^{2-s})} \\ &= 1 + \frac{2}{2^s} + \frac{16}{3^s} + \frac{36}{5^s} + \frac{32}{6^s} + \frac{64}{7^s} + \frac{168}{9^s} + \frac{72}{10^s} + \frac{144}{11^s} + \frac{196}{13^s} + \frac{128}{14^s} + \frac{576}{15^s} + \frac{324}{17^s} + \cdots . \end{aligned} \tag{9}$$

The CSLs split into two classes only, one of which has odd and the other even index. Hence, the generating function for the number of CSLs reads

$$\begin{aligned}\Phi_{\mathbb{Z}^4}(s) &= (1+2^{-s})\Phi_{D_4}(s) = (1+2^{-s})\prod_{p\neq 2}\frac{1+p^{-s}+2p^{1-s}+2p^{-2s}+p^{1-2s}+p^{1-3s}}{(1-p^{2-s})(1-p^{1-2s})}\\ &= 1+\frac{1}{2^s}+\frac{16}{3^s}+\frac{36}{5^s}+\frac{16}{6^s}+\frac{64}{7^s}+\frac{152}{9^s}+\frac{36}{10^s}+\frac{144}{11^s}+\frac{196}{13^s}+\frac{64}{14^s}+\frac{576}{15^s}+\frac{324}{17^s}+\cdots .\end{aligned}$$

Differences to Eq. (9) occur for all even integers and all integers that are divisible by the square of an odd prime. The spectrum is $\sigma = \{2^\ell m \mid \ell\in\{0,1\}, \text{and } m\in\mathbb{N} \text{ odd}\}$.

3 Root lattice A_4

Usually, the A_4 lattice — we will denote it by L in the following — is embedded in $\mathbb{R}^5$ as a lattice plane. However, this is inconvenient for our purposes and we prefer to look at it in $\mathbb{R}^4$, since we want to exploit the useful parameterisation by quaternions, which we do not have at hand in 5 dimensions. A possible basis for L consists of the 4 vectors

$$(1,0,0,0), \frac{1}{2}(-1,1,1,1), (0,-1,0,0), \frac{1}{2}(0,1,\tau-1,-\tau), \tag{10}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden mean whose algebraic conjugate τ' can be written as $\tau' = -\frac{1}{\tau} = 1-\tau$. The lattice L cannot be identified with a ring of quaternions. However, if we interpret the basis vectors as quaternions, they relate to the icosian ring $\mathbb{I}$, which is the $\mathbb{Z}[\tau]$–span of the 4 quaternions

$$(1,0,0,0), (0,1,0,0), \frac{1}{2}(1,1,1,1), \frac{1}{2}(1-\tau,\tau,0,1). \tag{11}$$

Note that neither L nor $\mathbb{I}$ are invariant under algebraic conjugation. Combining the algebraic conjugation with a permutation of the last two components yields an involution of the second kind $\tilde{x} := (x_0', x_1', x_3', x_2')$, which was called twist map in [12, 13]. Note that $L = \tilde{L}$ is invariant under the twist map, which in addition is an antiautomorphism of $\mathbb{I}$. The twist map is the key to our analysis since it provides us with a useful representation of L,

$$L = \{x\in\mathbb{I} \mid x = \tilde{x}\} = \{x+\tilde{x} \mid x\in\mathbb{I}\} \tag{12}$$

and its CSLs — see below.

Any coincidence rotation of L (and only those) can be parameterised by a single primitive admissible quaternion $q\in\mathbb{I}$ by means of $R(q) = \frac{1}{|q\tilde{q}|}qx\tilde{q}$ (see [13]). Here, admissible means that $|q\tilde{q}|\in\mathbb{N}$.

Admissibility guarantees that one can define the factors

$$\alpha_q := \sqrt{\frac{|\tilde{q}|^2}{\gcd(|q|^2,|\tilde{q}|^2)}} \qquad \text{and} \qquad \alpha_{\tilde{q}} := \alpha_q' = \sqrt{\frac{|q|^2}{\gcd(|q|^2,|\tilde{q}|^2)}}, \tag{13}$$

which are elements of $\mathbb{Z}[\tau]$. Note, however, that they are only defined up to units of $\mathbb{Z}[\tau]$. In addition, we define the extension of a primitive admissible quaternion by $q_\alpha = \alpha_q q$. The coincidence index now reads [13]

$$\varSigma_{A_4}(R(q)) = \frac{|q\tilde{q}|^2}{\gcd(|q|^2,|\tilde{q}|^2)} = |q\tilde{q}|\alpha_q\alpha_{\tilde{q}} = \mathrm{lcm}(|q|^2,|\tilde{q}|^2). \tag{14}$$

Consequently, the coincidence spectrum is $\mathbb{N}$. An explicit expression for the CSLs exists, too:

$$L\cap\frac{q_\alpha L\tilde{q}_\alpha}{|q_\alpha\tilde{q}_\alpha|} = \{q_\alpha x + \tilde{x}\tilde{q}_\alpha \mid x\in\mathbb{I}\} = (q_\alpha\mathbb{I}+\mathbb{I}\tilde{q}_\alpha)\cap L. \tag{15}$$

Along the same lines as for the hypercubic lattices, one can calculate the number of coincidence rotations and the number of CSLs of a given index n. The expressions obtained are slightly more complicated, since one has to distinguish between the three cases $p=5$, $p=\pm1 \pmod 5$ and $p=\pm2 \pmod 5$, as these

primes behave differently and correspond to the ramifying, splitting and inert primes of $\mathbb{Z}[\tau]$, respectively. If $120 f_{A_4}^{rot}(n)$ is the number of coincidence rotations — recall that the point group of L contains 120 elements — its generating Dirichlet series is given by

$$\begin{aligned}\Phi_{A_4}^{rot}(s) &= \frac{1+5^{1-s}}{1-5^{2-s}} \prod_{p\equiv\pm1(5)} \frac{(1+p^{-s})(1+p^{1-s})}{(1-p^{1-s})(1-p^{2-s})} \prod_{p\equiv\pm2(5)} \frac{1+p^{-s}}{1-p^{2-s}} \\ &= 1+\frac{5}{2^s}+\frac{10}{3^s}+\frac{20}{4^s}+\frac{30}{5^s}+\frac{50}{6^s}+\frac{50}{7^s}+\frac{80}{8^s}+\frac{90}{9^s}+\frac{150}{10^s}+\frac{144}{11^s}+\cdots .\end{aligned}$$

Similarly, we have for the number of CSLs

$$\begin{aligned}\Phi_{A_4}(s) &= \left(1+6\frac{5^{-s}}{1-5^{2-s}}\right) \prod_{p\equiv\pm2(5)} \frac{1+p^{-s}}{1-p^{2-s}} \prod_{p\equiv\pm1(5)} \frac{1+p^{-s}+2p^{1-s}+2p^{-2s}+p^{1-2s}+p^{1-3s}}{(1-p^{2-s})(1-p^{1-2s})} \\ &= 1+\frac{5}{2^s}+\frac{10}{3^s}+\frac{20}{4^s}+\frac{6}{5^s}+\frac{50}{6^s}+\frac{50}{7^s}+\frac{80}{8^s}+\frac{90}{9^s}+\frac{30}{10^s}+\frac{144}{11^s}+\cdots .\end{aligned}$$

Here, a criterion analogous to Theorem 1 exists, which will be published elsewhere [11].

4 Icosian ring

It is also interesting to discuss the coincidences of the icosian ring $\mathbb{I}$ itself. Note that it is no lattice in 4–space but a $\mathbb{Z}$–module of rank 8. However, this does not matter and one can argue similarly to the hypercubic case. The main difference is that one has to work with $\mathbb{Z}[\tau]$ instead of $\mathbb{Z}$, which does not cause any problems. In fact, some steps are even easier since the prime 2 does not play a special role here. So, a primitive quaternion is already reduced and hence the notion of reducibility is not needed here.

Let us call a pair $(q,p) \in \mathbb{I}\times\mathbb{I}$ *primitive admissible* if q,p are primitive and $|qp| \in \mathbb{Z}[\tau]$. Then, a rotation is a coincidence rotation if and only if it is parameterised by a primitive admissible pair (q,p). For any primitive admissible pair (q,p), we can once more define

$$\alpha_q := \sqrt{\frac{|p|^2}{\gcd(|q|^2,|p|^2)}} \qquad \text{and} \qquad \alpha_p := \sqrt{\frac{|q|^2}{\gcd(|q|^2,|p|^2)}},$$

which are again elements of $\mathbb{Z}[\tau]$, defined up to units. As before, $(q_\alpha,p_\alpha) := (\alpha_q q, \alpha_p p)$ is the corresponding extension pair. We get the following expression for the coincidence index [11]:

$$\Sigma_{\mathbb{I}}(R(q,p)) = \mathrm{N}(\mathrm{lcm}(|q|^2,|p|^2)) = \mathrm{N}(|q_\alpha|^2) = \mathrm{N}(|p_\alpha|^2).$$

Here $\mathrm{N}(a) = |aa'|$ is the (number theoretic) norm of $a \in \mathbb{Z}[\tau]$. Thus, the coincidence spectrum consists of all integers that contain prime factors $p = \pm 2 \pmod 5$ only with even power. The CSMs read explicitly

$$\mathbb{I} \cap \frac{q\mathbb{I}\bar{p}}{|qp|} = q_\alpha \mathbb{I} + \mathbb{I}\bar{p}_\alpha .$$

If $7200 f_{\mathbb{I}}^{rot}(n)$ denotes the number of coincidence rotations and $f_{\mathbb{I}}(n)$ the number of CSMs, we get again $f_{\mathbb{I}}^{rot}(n) \geq f_{\mathbb{I}}(n)$. Again, equality fails to hold in general, and an analogue of Theorem 1 tells us which CSMs are equal [11]. The corresponding Dirichlet series read

$$\begin{aligned}\Phi_{\mathbb{I}}^{rot}(s) &= \frac{(1+5^{-s})(1+5^{1-s})}{(1-5^{1-s})(1-5^{2-s})} \prod_{p\equiv\pm1(5)} \left(\frac{(1+p^{-s})(1+p^{1-s})}{(1-p^{1-s})(1-p^{2-s})}\right)^2 \prod_{p\equiv\pm2(5)} \frac{(1+p^{-2s})(1+p^{2-2s})}{(1-p^{2-2s})(1-p^{4-2s})} \\ &= 1+\frac{25}{4^s}+\frac{36}{5^s}+\frac{100}{9^s}+\frac{288}{11^s}+\frac{440}{16^s}+\frac{400}{19^s}+\frac{900}{20^s}+\frac{960}{25^s}+\cdots \end{aligned} \tag{16}$$

and

$$\Phi_{\mathbb{I}}(s) = \frac{1+5^{-s}+2\cdot 5^{1-s}+2\cdot 5^{-2s}+5^{1-2s}+5^{1-3s}}{(1-5^{2-s})(1-5^{1-2s})} \prod_{p\equiv\pm 1(5)} \frac{(1+p^{-s}+2p^{1-s}+2p^{-2s}+p^{1-2s}+p^{1-3s})^2}{(1-p^{2-s})(1-p^{1-2s})}$$
$$\times \prod_{p\equiv\pm 2(5)} \frac{1+p^{-2s}+2p^{2-2s}+2p^{-4s}+p^{2-4s}+p^{2-6s}}{(1-p^{4-2s})(1-p^{2-4s})}$$
$$= 1+\frac{25}{4^s}+\frac{36}{5^s}+\frac{100}{9^s}+\frac{288}{11^s}+\frac{410}{16^s}+\frac{400}{19^s}+\frac{900}{20^s}+\frac{912}{25^s}+\cdots . \tag{17}$$

Acknowledgements

The authors are grateful to Uwe Grimm, Manuela Heuer and Robert V. Moody for helpful discussions on the present subject. This work was supported by the German Research Council (DFG), within the CRC 701.

References

[1] M. Baake. Solution of the coincidence problem in dimensions $d \leq 4$. In R. V. Moody, editor, *The Mathematics of Long-Range Aperiodic Order*, pages 9–44, Dordrecht, 1997. Kluwer. Rev. version: `arXiv:math.MG/0605222`.

[2] M. Baake and U. Grimm. Multiple planar coincidences with N–fold symmetry. *Z. Kristallogr.*, 221:571–581, 2006. `arXiv:math.MG/0511306`.

[3] P. Zeiner. Multiple CSLs for cubic lattices. in preparation.

[4] M. Baake and P. Zeiner. Multiple coincidences in dimensions $d \leq 3$. *Phil. Mag.*, 87:2869–2876, 2007.

[5] P. A. B. Pleasants, M. Baake, and J. Roth. Planar coincidences for N–fold symmetry. *J. Math. Phys.*, 37:1029–1058, 1996. rev. version: `arXiv:math/0511147`.

[6] M. Koecher and R. Remmert. Hamilton's quaternions. In H.-D. Ebbinghaus et al., editors, *Numbers*, pages 189–220, Berlin, 1991. Springer.

[7] P. du Val. *Homographies, Quaternions and Rotations.* Clarendon Press, Oxford, 1964.

[8] M.-F. Vignéras. *Arithmétique des Algèbres de Quaternions.* Springer, Berlin, 1980.

[9] P. Zeiner. Coincidences of hypercubic lattices in 4 dimensions. *Z. Kristallogr.*, 221:105–114, 2006. `arXiv:math/0605526`.

[10] M. Baake, P. A. B. Pleasants, and U. Rehmann. Coincidence site modules in 3–space, 2007. `arXiv:math/0609793`.

[11] M. Baake and P. Zeiner. in preparation.

[12] M. Baake, M. Heuer, and R. V. Moody. Similar sublattices of the root lattice A_4. `arXiv:math/0702448`.

[13] M. Baake, M. Heuer, U. Grimm, and P. Zeiner. Coincidence rotations of the root lattice A_4. *Europ. J. Combinatorics*, in press. `arXiv:0709.1341`.